A Two-Zero Filter

Filter 'equation'

\[ y[n] = a_0 x[n] + a_1 x[n - 1] + a_2 x[n - 2] \]

\textit{z}-transform:

\[ Y(z) = a_0 X(z) + a_1 z^{-1} X(z) + a_2 z^{-2} X(z) \]

Transfer function:

\[ H(z) = \frac{Y(z)}{X(z)} = a_0 + a_1 z^{-1} + a_2 z^{-2} \]

To calculate frequency and phase response, evaluate:

\[ H(z) \bigg|_{z = e^{j\omega}} \]

If \( z^{-1} = -\frac{a_1 \pm \sqrt{a_1^2 - 4a_2a_0}}{2a_2} \) yields a pair of complex conjugate roots

\[ a_0 + a_1 z^{-1} + a_2 z^{-2} = (1 - Re^{j\theta} z^{-1})(1 - Re^{-j\theta} z^{-1}) = 0 \]

\[ = 1 - R(e^{j\theta} + e^{-j\theta})z^{-1} + R^2 z^{-2} \]

\[ = 1 - 2R \cos \theta z^{-1} + R^2 z^{-2} \]

then \( a_2 = R^2a_0 \rightarrow R = \sqrt{a_2/a_0} \)

and \( a_1 = -2a_0R \cos \theta \rightarrow \theta = \cos^{-1} \left( \frac{a_1}{-2a_0R} \right) \)
To calculate frequency and phase response, evaluate:

\[ H(z) \big|_{z = e^{j\hat{\omega}}} \]

\[
H(e^{j\hat{\omega}}) = a_0 + a_1 e^{-j\hat{\omega}} + a_2 e^{-2j\hat{\omega}}
\]

\[ = a_0 + a_1(\cos \hat{\omega} - j \sin \hat{\omega}) + a_2(\cos 2\hat{\omega} - j \sin 2\hat{\omega}) \]

\[ = (a_0 + a_1 \cos \hat{\omega} + a_2 \cos 2\hat{\omega}) + j(-a_1 \sin \hat{\omega} - a_2 \sin 2\hat{\omega}) \]

\[ = x + iy \]

where

\[ x = a_0 + a_1 \cos \hat{\omega} + a_2 \cos 2\hat{\omega} \]

and

\[ y = -a_1 \sin \hat{\omega} - a_2 \sin 2\hat{\omega} \]

Then frequency response:

\[ |H(e^{j\hat{\omega}})| = \sqrt{x^2 + y^2} \]

and phase response:

\[ \angle H(e^{j\hat{\omega}}) = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{-\left(a_1 \sin \hat{\omega} + a_2 \sin 2\hat{\omega}\right)}{a_0 + a_1 \cos \hat{\omega} + a_2 \cos 2\hat{\omega}} \]
**M-Zero Filter**

Filter ‘equation’

\[ y[n] = a_0 x[n] + a_1 x[n - 1] + \cdots + a_M x[n - M] \]

z-transform:

\[ Y(z) = a_0 X(z) + a_1 z^{-1} X(z) + \cdots + a_M z^{-M} X(z) \]

Transfer function:

\[ H(z) = \frac{Y(z)}{X(z)} = a_0 + a_1 z^{-1} + \cdots + a_M z^{-M} \]

To calculate frequency and phase response, evaluate:

\[ H(z) \big|_{z = e^{j\hat{\omega}}} \]

We can generalize the 2-Zero case to get

\[ H(e^{j\hat{\omega}}) = a_0 + a_1 e^{-j\hat{\omega}} + \cdots + a_M e^{-jM\hat{\omega}} \]

\[ = a_0 + a_1 (\cos \hat{\omega} - j \sin \hat{\omega}) + \cdots + a_M (\cos M\hat{\omega} - j \sin M\hat{\omega}) \]

\[ = (a_0 + a_1 \cos \hat{\omega} + \cdots + a_M \cos M\hat{\omega}) + j(-a_1 \sin \hat{\omega} - \cdots) \]

\[ = x + jy, \quad x = \sum_{k=0}^{M} a_k \cos k\hat{\omega} \text{ and } y = -\sum_{k=1}^{M} a_k \sin k\hat{\omega} \]

Gain and Phase:

\[ |H(e^{j\hat{\omega}})| = \sqrt{x^2 + y^2}; \quad \Theta = \tan^{-1} \frac{y}{x} \]
This is the most general case of filter we can discuss.

Filter ‘equation’

\[ y[n] = a_0 x[n] + \cdots + a_M x[n - M] - b_1 y[n - 1] - \cdots - b_1 y[n - N] \]

Transfer function:

\[
H(z) = \frac{Y(z)}{X(z)} = \frac{a_0 + \cdots + a_M z^{-M}}{1 + b_1 z^{-1} + \cdots + b_N z^{-N}}
\]

Proceeding as we did in the N-Zero case

\[
H(e^{j\hat{\omega}}) = \frac{a_0 + a_1 e^{-j\hat{\omega}} + \cdots + a_M e^{-jM\hat{\omega}}}{1 + b_1 e^{-j\hat{\omega}} + \cdots + b_N e^{-jN\hat{\omega}}}
\]

\[
= \frac{x_z + jy_z}{x_p + jy_p}
\]

where \( x_z = \sum_{k=0}^{M} a_k \cos k\hat{\omega}, y_z = -\sum_{k=1}^{M} a_k \sin k\hat{\omega} \)

\( x_p = \sum_{i=0}^{N} a_i \cos k\hat{\omega} \) and \( y_p = -\sum_{i=1}^{N} a_i \sin k\hat{\omega} \)
Since the magnitude of a ratio of two complex numbers is the ratio of the magnitudes and the argument of the ratio is the difference of the arguments

$$|H(e^{j\hat{\omega}})| = \frac{\sqrt{x_z^2 + y_z^2}}{\sqrt{x_p^2 + y_p^2}}$$

$$\Theta = \tan^{-1} \frac{y_z}{x_z} - \tan^{-1} \frac{y_p}{x_p}$$

So, if a given filter equation cannot be simplified nicely along the lines of our very first, 1-zero example, by knowing the coefficients $a_i$ and $b_i$ we can calculate the gain and phase response at an angle $\hat{\omega}$.

The algorithm in the perl code `filt-resp.pl` simply considers $k$ different angles in the region $0 \ldots \pi$ and evaluates $|H(e^{j\hat{\omega}})|$ and $\Theta$ above at each value.
• To avoid distortion, audio filters should normally have linear phase characteristics

• Normally start with a specification of frequency behaviour of a filter and the problem becomes one of determining the system equation $a_i$ and $b_i$ that’ll deliver the required performance within a specified tolerance limit

• We saw earlier that an arbitrary system can be “factored” into several simpler ones that are cascaded together; this occurs quite frequently in practice