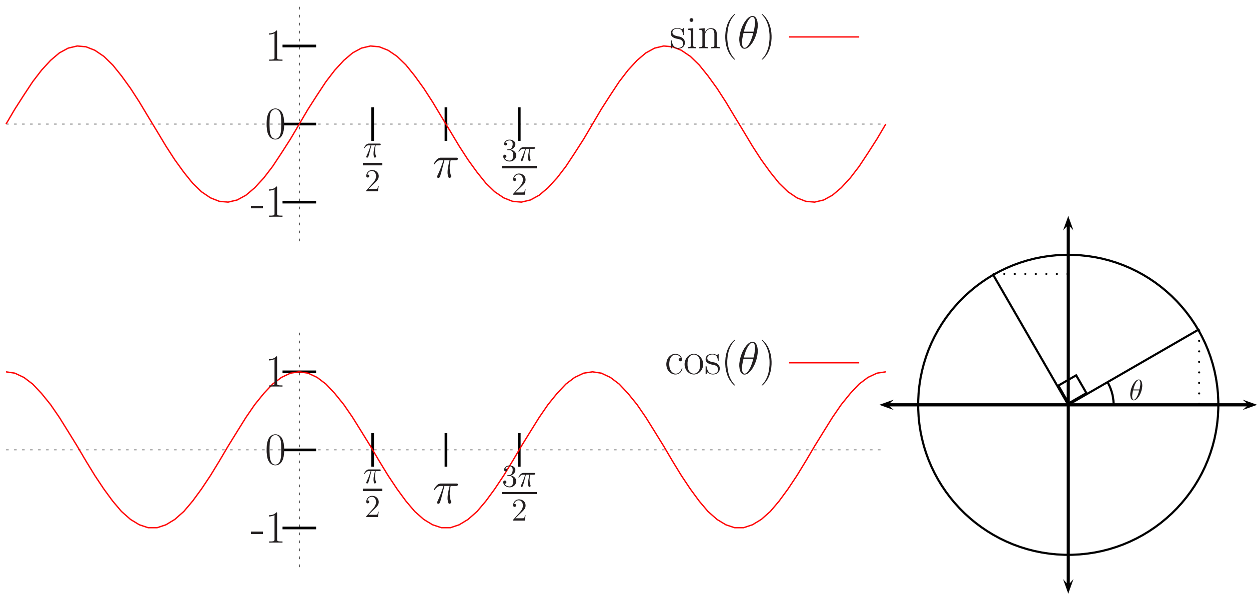
Phase Shift

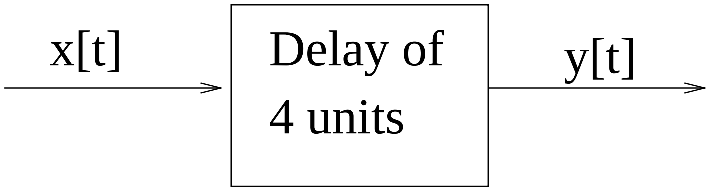
- Given a function $f(t)$, what does $g(t) = f(t - a)$ look like?
- For a value of t, $g(t)$ has the same value as the function f had a units earlier
- That is, $g(t)$ is a *right* shifting of $f(t)$
- Similarly, if $g(t) = f(t+a)$, then $g(t)$ is $f(t)$ shifted *left* by a units



From the plots, $g(\theta) = \cos(\theta)$ is the function $f(\theta) = \sin(\theta)$ shifted *left* by $\frac{\pi}{2}$ radians. So $\cos(\theta) = \sin(\theta + \frac{\pi}{2})$.

Similarly, $g(\theta) = \sin(\theta)$ is the function $f(\theta) = \cos(\theta)$ shifted *right* by $\frac{\pi}{2}$ radians. So $\sin(\theta) = \cos(\theta - \frac{\pi}{2})$.

Time Delay System



- If some discrete (sampled) signal, $x[t]$ is passed through a "delay of 4" system, then the output, $y[t] = x[t - 4]$

Sinusoidal Signals

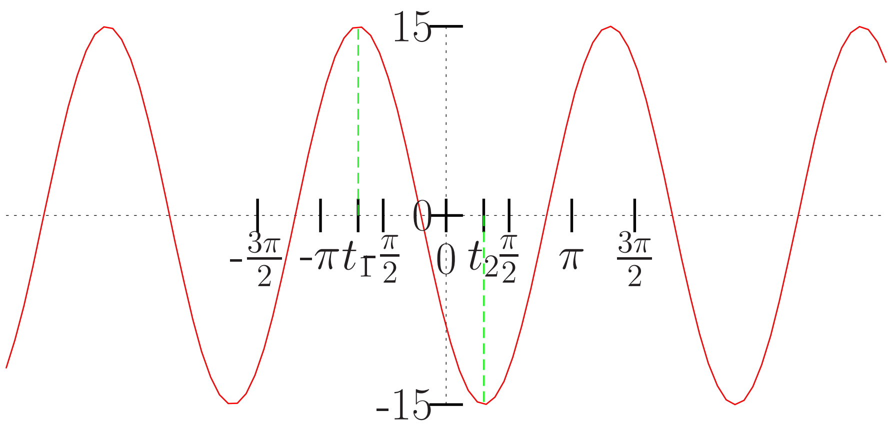
- Almost all of the inputs we look at in the course will be made up of sinusoidal or combinations of sinusoidal signals
- The most general form

$$x(t) = A\cos(\omega_0 t + \phi)$$

- This can be thought of as asking what is the x component of the position of an arm rotating c'clockwise
- A is the magnitude and is the very largest value that the signal can reach
- ϕ is a *phase shift*; an interpretation of ϕ is that instead of the arm rotating c'clockwise starting from 0 radians ("three o'clock"), the arm is at an angle of ϕ at time $t = 0$
- ω_0 is the *radian frequency* (speed of rotation of the arm); since it's a speed, the units must be radians/sec
- Note: in some books sinusoids are expressed in terms of sin, rather than cos; since we know that $\cos(\theta) = \sin(\theta + \frac{\pi}{2})$ the only difference between the two is a difference of phase

Sinusoidal Signals (contd.)

$15\cos(t + 0.7 * \pi)$



$$t_1 = -0.7\pi; \qquad t_2 = 0.3\pi$$

Frequency and Period of a Signal

We call ω_0 the angular speed of the signal; clearly, the faster the arm rotates, the more peaks and troughs there will be in a given time window

We define the *frequency*, f_0, of the signal to be

$$\boxed{f_0 = \frac{\omega_0}{2\pi}}$$

This is the number of revolutions that the arm makes in a unit of time.

In tandem with how often the signal repeats, we may also wonder what is the length of time before a signal with angular speed ω_0 starts repeating; this is called the *period*, T_0, of the signal.

That is, when does the following equality hold?

Sinusoidal Signals (contd.)

$$x(t + T_0) = x(t)$$

$$A\cos(\omega_0(t + T_0) + \phi) = A\cos(\omega_0 t + \phi)$$

$$\cos(\omega_0 t + \omega_0 T_0 + \phi) = \cos(\omega_0 t + \phi)$$

Since the cos function has a period of 2π, the above holds if $\omega_0 T_0 = 2\pi$ or

$$\boxed{T_0 = \frac{2\pi}{\omega_0}}$$

Or, using the relationship of frequency, $2\pi f_0 = \omega_0$

$$\boxed{T_0 = \frac{1}{f_0}}$$

Sinusoidal Signals as Phasors

- We will spend much of the semester looking at sinusoidal signals so it is important that we be able to represent them in a way that is easy to work with
- One common way of representing a sinusoid is by a *phasor* – a stuffy name for our rotating arm
- We will look at two ways of representing a sinusoid in terms of phasors

Sinusoidal Signals as Phasors (contd.)

- We have seen how we can describe a complex number z as $z = re^{j\theta}$
- We can do the same for a *complex* signal by saying

$$z(t) = Ae^{j(\omega_0 t+\phi)}$$

- Then

$$z(t) = A\cos(\omega_0 t + \phi) + jA\sin(\omega_0 t + \phi) = x(t) + jy(t)$$

and so,

$$\boxed{x(t) = \Re\{z(t)\}}$$

- That is, a sinusoid can be found from looking at the x component of a rotating phasor that starts (time $t = 0$) at angle ϕ and rotates at angular speed ω_0 radians/sec
- Alternatively, we can express the sinusoid as the sum of *two* phasors, each of half the magnitude of the sinusoid, that rotate in opposite directions, but each with angular speed ω_0
- To do this we will give a justification for the identity

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

(We could show similarly that $\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$.)

- We have seen the following power series expansion for e^x and e^{-x}

$$e(x) = 1 + x + \frac{1}{2!}x^2 + \cdots$$

$$e(-x) = 1 - x + \frac{1}{2!}x^2 - \cdots$$

$$\frac{e^{jx} + e^{-jx}}{2} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots$$

$$= \cos(x)$$

- So

$$A\cos(\omega_0 t + \phi) = A\Big(\frac{e^{j(\omega_0 t+\phi)} + e^{-j(\omega_0 t+\phi)}}{2}\Big)$$

$$= \frac{A}{2}e^{j\phi}e^{j\omega_0 t} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 t}$$

- Two phasors, each of magnitude $\frac{A}{2}$ and angular speed ω_0, one rotating counterclockwise and the other rotating clockwise sum together to give the real (x or cos) part of a phasor with magnitude A and same angular frequency