Complex Numbers (contd.)

Arithmetic: which co-ords?

- We can only do multiplication and division when complex numbers are expressed in polar form
- If we want to add to subtract two numbers in polar form, we must convert them to rectangular first
- OTOH, we can do all four arith. ops in rectangular form
- *In spite of this*, polar form is the preferred form to think about complex numbers because it tells us the magnitude, $|z| = \sqrt{x^2 + y^2}$ of the number

Roots of unity

- It is easy to take powers in polar form, also
- $z^N = (re^{j\theta})^N = r^N e^{jN\theta}$
- Since $e^{j\theta} = \cos\theta + j\sin\theta$, we get de Moivre's famous theorem

$$(\cos\theta + j\sin\theta)^N = \cos N\theta + j\sin N\theta$$

- When we consider the important Discrete Fourier Transform (DFT), we will need to know the Nth roots of unity
- That is, if $z^N = 1$, what is z?

Complex Numbers (contd.)

- There will be N solutions to $Z^N - 1 = 0$ and these are given by

$$z = e^{j2\pi l/N}, \quad l = 0, 1, \ldots, N-1$$

- Note that $z' = ze^{j2\pi}$ will also be a solution because $(z')^N = z^N e^{j2N\pi} = z^N$
- So, adding (or subtracting) 2π gives another solution, but since $\theta + 2\pi$ is not a principal angle, these solutions are not interesting
- OTOH, the multiples of 2π in the *exponent* are important, since they are being divided by N and each value of l will give an angle somewhere in $[0, 2\pi]$

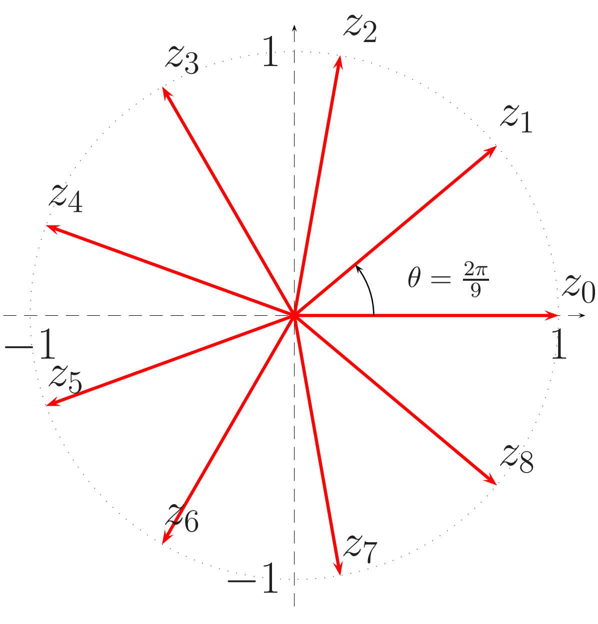
Solve $z^9 = 1$

Basic solution is $z = e^{j2\pi l/9}, \quad l = 0, 1, \ldots, N-1$

For any $0 \le l \le 8$, $z = e^{j2\pi l/9}$ is a solution

Consider $z_5 = e^{j2\pi 5/9}$; that is, $l = 5$

$$z_5^9 = (e^{j2\pi 5/9})^9 = e^{j2\pi 5} = e^{j10\pi} = e^{j0} = 1$$



Complex Numbers (contd.)

Roots of a complex number c

- How do we solve $z^N = c$, where c is a complex number?
- Write c in polar form as $c = |c|e^{j\phi}$
- Generalize c to take account of division by N that comes later: $c = |c|e^{j\phi}e^{j2\pi l}, \forall l \in \mathbb{N}$
- Write solution z as $z = |z|e^{j\theta}$, and so

$$|z|^N e^{jN\theta} = |c|e^{j\phi}e^{j2\pi l}$$

- So $|z| = |c|^{\frac{1}{N}}$ and $e^{jN\theta} = e^{j\phi + j2\pi}$; so

$$N\theta = \phi + 2\pi l$$

$$\theta = \frac{\phi + 2\pi l}{N}$$

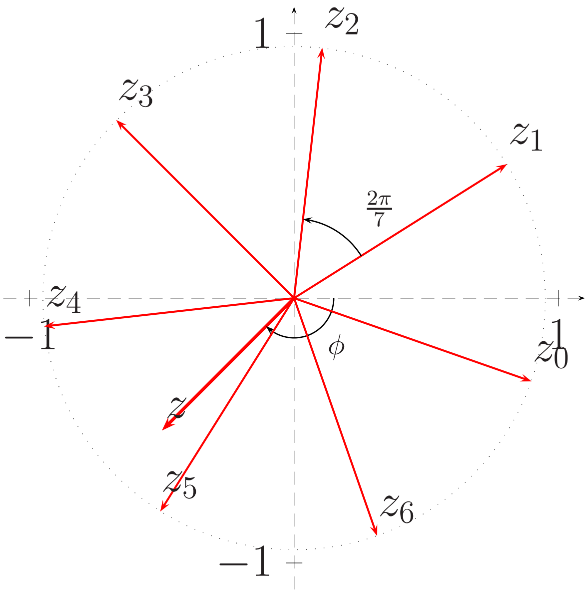
$$= \frac{\phi}{N} + \frac{2\pi l}{N}$$

- Note 1: the first term is independent of l and once $l \geq N$, $2\pi l/N \geq 2\pi$ so we begin to see repeats of solutions when $l \leq N$
- Note 2: If $c = 1$, $c = 1e^{j0}$ so $\phi = 0$ and $|z| = 1^{\frac{1}{N}} = 1$

The 7^{th} roots of $z = (-\frac{1}{2}, -j\frac{1}{2})$

The principal root is $z_0 = |z|^{\frac{1}{7}} \angle \frac{1}{7}\phi$, where $\phi = \angle z$. With respect to this principal root, raising any of the others to the 7^{th} power (stepping around 7 times) will bring you back to the starting value, z.

Note that $|z| = 0.7071$ and $\phi = \angle z = -5\pi/4$; so, $|z|^{\frac{1}{7}} = 0.9517$.



The 5^{th} roots of $z = (-\frac{3}{2}, j)$

The principal root is $z_0 = |z|^{\frac{1}{5}} \angle \frac{1}{5}\phi$, where $\phi = \angle z$. With respect to this principal root, raising any of the others to the 5^{th} power (stepping around 5 times) will bring you back to the starting value, z.

Note that $|z| = 1.802776$ and $\phi = \angle z = 2.553590$; so, $|z|^{\frac{1}{5}} = 1.125093$ and $\phi/5 = 0.510718$.