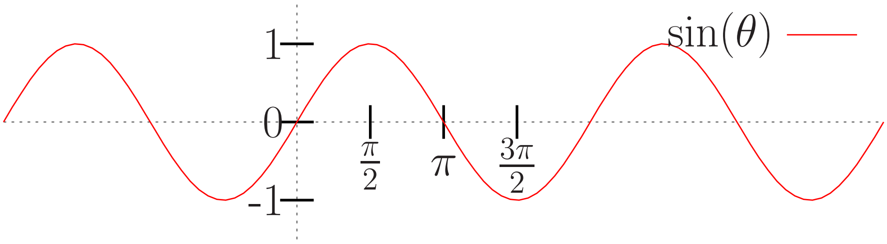
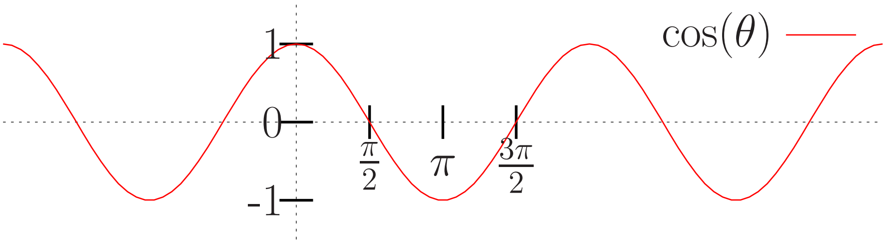
Complex Numbers (contd.)





- By considering the *slopes* of the above two functions, we can discern that
  - if $f(x) = \sin(x)$ then $f'(x) = \cos(x)$
  - if $f(x) = \cos(x)$ then $f'(x) = -\sin(x)$
- This is of use because, using a Maclaurin Series, we can express $\cos x$ or $\sin x$ as a **power series**

  Maclaurin Series If a function, $f(x)$, can be differentiated as often as you like, then it can be expressed as the following power series

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(i)}(0)}{i!}x^i + \cdots$$

- See the entry on *Maclaurin Series* (an easier, special case of Taylor Series) in Wikipedia for more info

Complex Numbers (contd.)

How does a MacLaurin Series work for $f(x) = \sin x$?

$f'(x) = \cos x$; $f''(x) = -\sin x$; $f'''(x) = -\cos x$; $f^{(4)}(x) = \sin x$

So,

$$\sin x = \sin 0 + \frac{x}{1!}\cos 0 - \frac{x^2}{2!}\sin 0 - \frac{x^3}{3!}\cos 0 + \frac{x^4}{4!}\sin 0 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$= \sum_{i=0}^{\infty}(-1)^i \frac{1}{(2i+1)!}x^{2i+1}$$

In the same way,

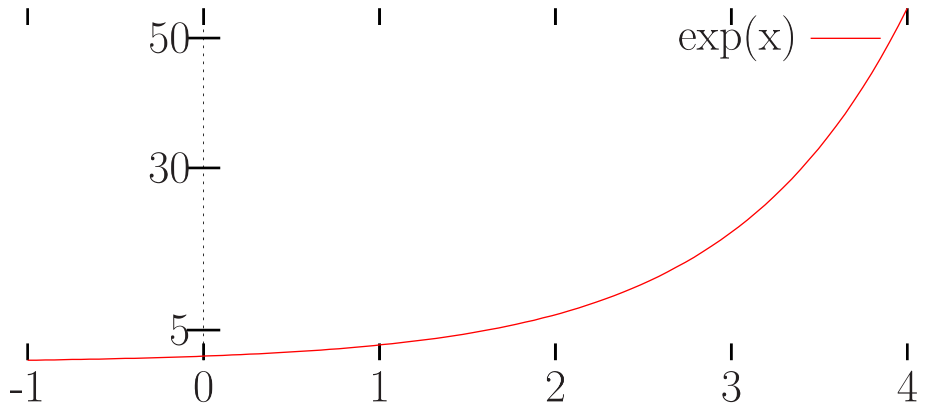
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$= \sum_{i=0}^{\infty}(-1)^i \frac{1}{(2i)!}x^{2i}$$

Complex Numbers (contd.)

The function e^x

- Functions that grow at an exponential rate have always been studied because they represent solutions to many "systems"
- $y = f(x) = a^x$ (x continuous) and $f(n) = a^n$ (n discrete) are two examples
- Many systems (e.g., the pendulum example in Chapter 2 of book) need the derivative of the function $y = f(x) = a^x$
- It is an amazing fact that if $f(x) = e^x$ then the function is its own derivative; $f'(x) = e^x = f''(x) = \cdots$



- And if the function is $y = f(x) = e^{ax}$ then $f'(x) = ae^{ax}$, which very closely resembles $f(x)$.

Complex Numbers (contd.)

- What is the MacLaurin series for $f(x) = e^x$?

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(i)}(0)}{i!}x^i + \cdots$$

- We can use $f^{(i)}(x)$ to denote the i^{th} derivative of $f(x)$
- We've seen that $f'(x) = e^x = f''(x) = \cdots = f^{(i)}(x), \forall i$
- So $f^{(i)}(0) = 1, \forall i$ and then

$$\begin{aligned} e^x &= e^0 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{i!}x^i + \cdots \\ &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^i}{i!} + \cdots \\ &= \sum_{i=0}^{\infty} \frac{x^i}{i!} \end{aligned}$$

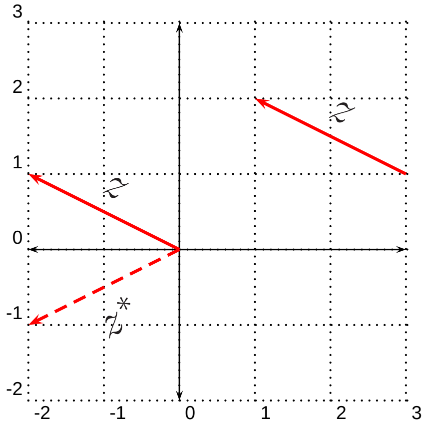
Complex Numbers (contd.)

Origin of complex numbers

- $\sqrt{-4} = \sqrt{4}\sqrt{-1} = 2i$
- Using quadratic formula to solve for x in $3x^2 - 3x + 5 = 0$:

$$x = \frac{3 \pm \sqrt{9 - 4.3.5}}{6} = \frac{1}{2} \pm \sqrt{-51} = \frac{1}{2} \pm \sqrt{51}i$$

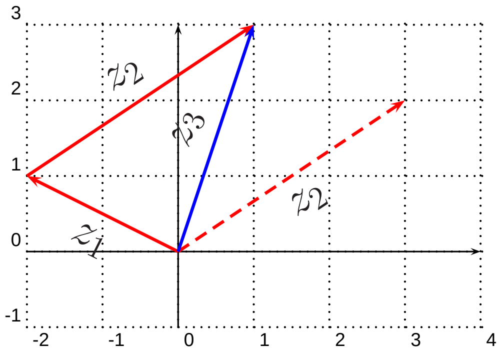
- For consistency with SPF book we'll use j instead of i from now on
- The complex no. $z = a + jb$ has *real part* a and *imaginary part* b
- We can plot a complex number in the (x, y) plane by letting x be the real part of z and y be the imaginary part
- Two representations of the complex number $z = -2 + j1$ are given below
- Also shown is the special complex number $z^* = a - jb$ for $z = -2 + j1$; this is called the *complex conjugate* of z



Complex Numbers (contd.)

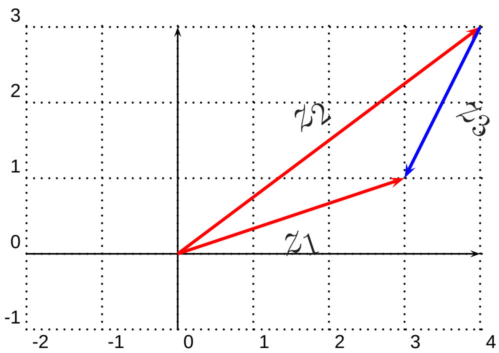
Addition

- To add two complex numbers we add the real parts together and the imaginary parts together
- If $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$ then $z_3 = z_1 + z_2 = (a_1 + a_2) + j(b_1 + b_2) = z_2 + z_1$
- Graphically, adding $z_1 = -2 + j$ and $z_2 = 3 + j2$

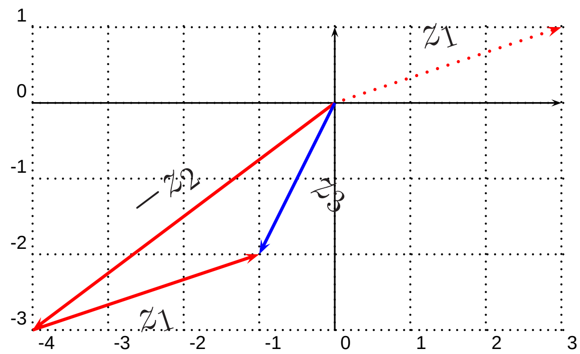


Subtraction

- To subtract two complex numbers we subtract the real parts and the imaginary parts
- If $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$ then $z_3 = z_1 - z_2 = (a_1 - a_2) + j(b_1 - b_2)$
- Graphically, subtracting $z_2 = 4+3j$ from $z_1 = 3+j1$ ($z_3 = z_1 - z_2$)



or



Multiplication

- The rule for multiplying $z_1 = a + bj$ and $z_2 = c + dj$ is identical to multiplying $(a + b)(c + d)$
- $z = z_1 z_2 = ac + bcj + adj - bd = (ac - bd) + (bc + ad)j$
- Let $z_1 = a + bj$. What is $z = z_1 z_1^*$? z_1^*, is the *complex conjugate* of z_1 is $a - bj$

$$
\begin{aligned}
z = z_1 z_1^* &= (a + bj) \times (a - bj) \\
&= a^2 + b^2 + (ab - ab)j \\
&= a^2 + b^2
\end{aligned}
$$

- Graphical interpretation to come later

Division

- To find $z = \frac{z_1}{z_2}$ we need z_2's complex conjugate, z_2^*

$$
\begin{aligned}
z = \frac{z_1}{z_2} = \frac{a + bj}{c + dj} &= \frac{(a + bj)z_2^*}{(c + dj)z_2^*} \\
&= \frac{(a + bj)(c - dj)}{c^2 + d^2} \\
&= \frac{(ac + bd) + (bc - ad)j}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} j
\end{aligned}
$$

Complex Numbers (contd.)

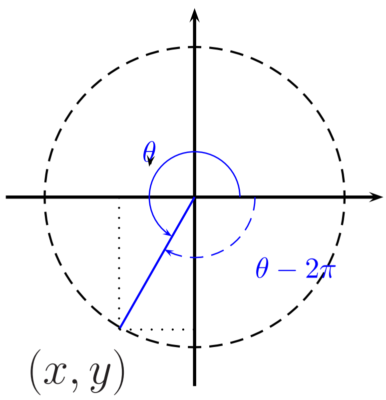
- If $z = x + jy$ what is the inverse of z, z^{-1}?
- $z^{-1}z = 1$, so $z^{-1} = \frac{1}{z}$
- And

$$z^{-1} = \frac{1}{z} = \frac{z^*}{zz^*} = \frac{z^*}{zz^*} = \frac{(x - jy)}{x^2 + y^2} = \frac{z^*}{x^2 + y^2}$$

Co-ordinates: Polar ↔ Cartesian

For the Cartesian (rectangular) point (x, y), the *polar* equivalent is $r\angle\theta$, where $r^2 = x^2 + y^2$ and $\tan\theta = y/x$

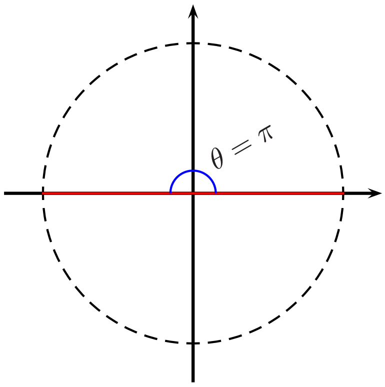
For the polar point $r\angle\theta$, the Cartesian point is $(x, y) = (r\cos\theta, r\sin\theta)$



Complex Numbers (contd.)

- If $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^i}{i!} + \cdots$, what is $e^{j\theta}$?

$$e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} + \cdots$$
$$= (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots) + j(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots)$$
$$= \cos\theta + j\sin\theta$$



From diagram, $\sin\pi = 0$

and $\cos\pi = -1$

So $e^{j\pi} = -1 + j0$

Descartes considered $e^{j\pi} + 1 = 0$ as proof of the existence of God!

- Using the above

$$re^{j\theta} = r\cos\theta + jr\sin\theta$$

But this is the representation of the complex no. $z = x + jy$ so, with r and θ defined as usual:

$$\boxed{z = x + jy = re^{j\theta}}$$

$$\boxed{z^* = x - jy = re^{-j\theta}}$$

- We can now do multiplication and division much more simply...

Complex Numbers (contd.)

- If $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$

$$z_1 z_2 = r_1 e^{j\theta_1} r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1+\theta_2)}$$

  and

$$\frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)}$$

- If $z = re^{j\theta}$, then $\boxed{z^* = re^{-j\theta}}$

- If $z = re^{j\theta}$, then either

$$z^{-1} = \frac{1}{z} = \frac{1e^{j0}}{re^{j\theta}} = \frac{1}{r}e^{-j\theta}$$

  or

$$z^{-1} = (re^{j\theta})^{-1} = r^{-1}(e^{j\theta})^{-1} = r^{-1}e^{-j\theta}$$

  But either way, we get

$$z^{-1} = \frac{1}{r^2} re^{-j\theta} = \frac{1}{r^2} z^*$$