Pre-Lab and Warm-Up: You should read at least the Pre-Lab and Warm-up sections of this lab assignment and go over all exercises in the Pre-Lab section before going to your assigned lab session.

Verification: The Warm-up section of each lab must be completed during your assigned Lab time and the steps marked Instructor Verification must also be signed off during the lab time. One of the laboratory instructors must verify the appropriate steps by signing on the Instructor Verification line. When you have completed a step that requires verification, simply demonstrate the step to the TA or instructor. Turn in the completed verification sheet to your TA when you leave the lab.

Lab Report: It is only necessary to turn in a report on Section 5 with graphs and explanations. You are asked to label the axes of your plots and include a title for every plot. In order to keep track of plots, include your plot inlined within your report. If you are unsure about what is expected, ask the TA who will grade your report.

1 Introduction & Objective

The goal of the laboratory project is to show that Fourier Series analysis is a powerful method for predicting the response of a system when the input is a periodic signal. Since we will be doing Fourier Series for continuous-time signals, the formulas are integrals. As a result we will use MATLAB’s Symbolic Toolbox which actually relies on MAPLE to do symbolic integration and symbolic algebra. Finally, we will use Fourier Series to analyze the same power supply problem as in Lab 14a, but now we can analyze the problem in the frequency domain.

2 Background: Fourier Series Analysis and Synthesis

Recall the analysis integral and synthesis summation for the Fourier Series expansion of a periodic signal $x(t) = x(t + T_0)$. The Fourier synthesis equation for a periodic signal $x(t) = x(t + T_0)$ is

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

(1)

where $\omega_0 = 2\pi/T_0$ is the fundamental frequency. To determine the Fourier series coefficients from a signal, we must evaluate the analysis integral for every integer value of $k$:

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)e^{-jk\omega_0 t} dt$$

(2)

where $T_0 = 2\pi/\omega_0$ is the fundamental period. If necessary, we can evaluate the analysis integral over any period; in (2) the choice $[-\frac{1}{2}T_0, \frac{1}{2}T_0]$ is sometimes a convenient one, but integrating over the interval $[0, T_0]$ would also give exactly the same answer.
3 Pre-Lab

In this project, we will use a feature of MATLAB that you may not have experienced. MATLAB has a symbolic toolbox that is based on the symbolic algebra program called MAPLE, which you might have used in calculus. With this toolbox we can evaluate mathematical expressions symbolically and then use MATLAB’s numerical functions and plotting functions to evaluate formulas and display results. We will use only a rudimentary set of the symbolic capabilities.

3.1 Symbolic Variables

You can define symbolic variables in several ways. The easiest is to simply type

```
syms v t A omega phi
```

Note that we use spaces instead of punctuation. This defines the variables v, t, A, omega, and phi as symbolic variables. Another way is to type `v=sym('v')`. Now when we use these variables they are interpreted as symbolic variables and we can make up symbolic mathematical expressions using the operators of MATLAB. For example the statements

```
syms v t A omega phi
v = A*cos(omega*t + phi)
```

would be used to define a “symbolic cosine wave”, which would be written in standard mathematical notation as

\[ v(t) = A \cos(\omega t + \phi). \]

Type `help whos` to see how the symbolic variable is stored in the workspace; also consult `help syms` to learn more.

3.2 Symbolic Integrals and Derivatives

MATLAB can do integration and differentiation with the functions `int()` and `diff()`. Verify that the following give the expected result from Calculus:

```
syms t A omega phi
xt = int(A*t^3 + pi, t)
yt = diff(A*cos(omega*t + phi), t)
zz = int(A*cos(omega*t + phi), t, 0, 1/omega)
zsimp = simple(zz)
```

Notice that you have to tell these functions which variable is the “variable of interest” for taking the derivative or doing the integral. The fourth line shows that you can also perform a definite integral by giving the limits of integration; and the limits can be symbolic. Furthermore, the last line shows that you might want to force MAPLE to simplify the formula.

Idea: you might speculate on using `int()` to do an integral that would arise in convolution such as:

\[ \int_{0}^{t} e^{-(t-\tau)} d\tau \]
3.3 Substituting Values

We generally want to substitute values for some of the symbolic variables, so that we can evaluate the functions that they represent. For example, we could obtain a symbolic cosine wave with amplitude $A=100$, $\omega=120\pi$, and $\phi=0$ by using the \texttt{subs( )} function:

$$ v1 = \text{subs}(v, \{A,\omega,\phi\}, \{100,120\pi,0\}) $$

which produces the new symbolic cosine wave $v1=100\cos(120\pi t)$. Consult \texttt{help subs}.

We can evaluate $v1$ which is a symbolic function of $t$ for 101 values covering one period, and then convert the values to MATLAB’s double precision floating-point form using the statements:

```matlab
tn = (0:0.01:1)/60;
vl = double( subs(v1,t,tn) ); %-- convert to numeric
plot(tn, vl)
```

Type \texttt{help double} to learn more about converting to the numerical format called double-precision.

Although we have used the \texttt{plot( )} function to plot these numeric values, there is another way that works much nicer for symbolic functions. This is \texttt{ezplot( )} which figures out the grid to use and frees the user from specifying the vectors $tn$ and $vl$ above. Consult \texttt{help ezplot} to learn more.

$$ \texttt{ezplot(v1, [0,1/60])} $$

4 Warmup

4.1 Symbolic Function for Fourier Synthesis

In this project we are going to use the symbolic features of MATLAB to determine Fourier series representations for periodic waveforms, synthesize the signals, and then plot them. In general, the limits on the sum in Eq. (1) are infinite, but for our computational purposes, we must restrict them to be a finite number $N$ obtaining the $2N+1$ term approximation

$$ x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}. \quad (3) $$

To illustrate how we can use symbolic math in MATLAB, consider the following function for evaluating (3):

```matlab
function xt = fouriersynth(ak, N, T0)
%FOURIERSYNTH synthesize Fourier Series formula SYMBOLICALLY
% usage: xt = fouriersynth(ak, N, T0)
% ak = (symbolic) formula for the Fourier Series coefficients
% N = (numeric) use the Fourier coeffs from -N to +N
% T0 = (numeric) Period in secs
% xt = (symbolic) signal synthesized
% example:
% syms ck k xt
% ck = sin(k)/k
% xt = fouriersynth(ck, 4, pi);
```

syms t k wwk
kk = -N:N;
try
    ak_num = subs( ak, k, kk+((kk==0)+sign(kk))*1e-9 )
catch
    error('FOURIERSYNTH: ak must use k as its variable')
end
wwk = exp(j*(2*pi*kk'/T0)*t);
xt = simple( ak_num*wwk );  %-- inner product

This function takes in one symbolic variable and two numeric variables: ak is a symbolic expression for the Fourier coefficients $a_k$ as a function of $k$; $N$ controls the number of Fourier coefficients to use in the synthesis, and $T_0$ specifies the numeric value of the period for the synthesized signal.

(a) You can download fouriersynth.m for later use, or you can type it into a file with that name. Study this program and explain what the last two lines do. Explain what is contained in the vector ak_num. Explain how this program creates a symbolic variable that is the same as the Fourier Synthesis summation formula (3).

(b) Here is an example of how we would use this function to synthesize a periodic function whose Fourier coefficients are given by $a_k = \sin(k)/k$ and whose period is 2.

\begin{verbatim}
syms wt t ak k
N = 10;  T0 = 2;
ak = sin(k)/k;
wt = fouriersynth(ak, N, T0);
ezplot(wt, [-To,T0]); grid on
axis tight  %---make ezplot show the whole thing
\end{verbatim}

Type in these instructions and observe the plot. Try increasing $N$ and running the program again. What happens as you increase $N$? What signal would result when $N \to \infty$? What is the DC value of the signal? From the plot? From the $a_k$ formula?

4.2 Symbolic Integration and Fourier Analysis

Using MAPLE, MATLAB can evaluate Fourier Series analysis integrals such as (2). Here is an example:

\begin{verbatim}
syms t xt XNt ak wwk k omega
T0 = 6;
xt = sin(2*pi*t/T0);
wwk = exp(-j*(2*pi*k/T0)*t)
ak = (1/T0)*int( xt*wwk, t, 0, T0/2)
ak = simple( ak )
N = 9;
xNt = fouriersynth( ak, N, T0);
ezplot(xNt, [-T0/2,2*T0]); grid on
\end{verbatim}
Note that we first obtain a symbolic expression $a_k$ for the coefficients by doing an integral followed by a simplification. Then we synthesize the signal.

Type in this program and run it. Give an equation valid over one period for the exact signal that is Fourier analyzed by the above integration. What is the fundamental period and the fundamental frequency? The semicolon ; is omitted from the expression for $a_k$ above, so the general expression for the Fourier coefficients will be typed out. Write the mathematical formula that is the general expression for $a_k$ in this case. Try larger or smaller values of $N$ to see the convergence of the approximate synthesis.

5 Lab Exercises: Analysis of a Power Supply Circuit

In this lab exercise, you will analyze the same power supply circuit as in Lab 14a, but the analysis will be done using symbolic representations. As in Lab 14a, the strategy consists of finding the Fourier Series for the input $x(t)$; multiplying by the frequency response of the lowpass filter; and then evaluating the size of the Fourier coefficients for the output signal. Figure 1 depicts the AC-to-DC voltage converter which consists of a diode bridge that implements a full-wave rectifier, followed by an RC circuit that provides lowpass filtering. The full-wave rectifier is defined by the equation

$$v(t) = \sqrt{2} \cos(120\pi t)$$

where $v(t)$ is the input and $x(t)$ is the output of the diode bridge rectifier. The output of the rectifier, $x(t)$, is the input to the RC circuit and $y(t)$ is the output of that circuit. Once you have learned circuits, it will be easy to write the differential equation that describes the R-C circuit in Fig. 1. The following differential equation gives the relationship between the input and output voltages of the circuit:

$$\frac{d}{dt} y(t) + \left(\frac{2200 + R}{2200RC}\right) y(t) = \frac{1}{RC} x(t)$$

In a real power supply, the signal $v(t)$ would be the 60-Hz powerline voltage, which would be represented mathematically as $v(t) = 110\sqrt{2} \cos(120\pi t)$. The units of $v(t)$ are volts.

The block diagram shown in Fig. 2 represents the operations performed by the different parts of the circuit. The input-output equation for the full-wave rectifier is defined by $x(t) = |v(t)|$. The purpose of the rectifier is to generate a periodic signal with a non-zero DC component. The purpose of the lowpass filter is to remove most of the high frequencies in the output of the rectifier, leaving the DC component.

5.1 Fourier Analysis of the Full-Wave Rectifier Output

Assume that the input is a power line voltage $v(t) = 110\sqrt{2} \cos(120\pi t)$. 

Instructor Verification (separate page)
Figure 2: Block diagram representation of power supply.

(a) It is not too hard to verify that the true impulse response of the R-C circuit is

\[ h(t) = \frac{1}{RC} e^{-\alpha t} u(t) \]

where \( \alpha = (2200 + R) \div 2200RC \). You don’t have to include this verification in a lab report, but it involves taking the first derivative and adding together the two terms on the left-hand side of the differential equation (5). Remember that the derivative of the unit-step signal, \( u(t) \), is the unit impulse signal, \( \delta(t) \).

Therefore, we can “take the Fourier transform” of \( h(t) \) and get the frequency response of the lowpass filter in the following form:

\[ H(j\omega) = \frac{\beta}{j\omega + \alpha} \]

Notice that the values of \( \alpha \) and \( \beta \) can be written in terms of \( R \) and \( C \); you will need this later on when evaluating the frequency response \( H(j\omega) \).

Write a function that plots of the magnitude and phase of \( H(j\omega) \) over the range \(-2\pi \leq \omega \leq 2\pi\). Write this code from scratch; don’t use a MATLAB built-in function.\(^1\) Test your function by making the plots for some typical values of \( R \) and \( C \) (e.g., use values from Lab 14a \( R = 33000 \) ohms and \( C = 5 \times 10^{-6} \) farads).

(b) Symbolic Toolbox: Make symbolic MATLAB expressions for the functions \( v(t) \) and \( x(t) \). The output of the full-wave rectifier is a periodic signal such that \( x(t) = x(t + T_0) \). Determine the fundamental period \( T_0 \) of the rectified signal \( x(t) \). Use \( subplot( ) \) and \( ezplot( ) \) to make a plot showing both \( v(t) \) and \( x(t) \) over the range \( 0 \leq t \leq 3T_0 \).

(c) Symbolic Toolbox: Since \( x(t) \) is a periodic signal, it has a Fourier Series. Write the MATLAB code to evaluate the symbolic Fourier coefficients of \( x(t) \). Make the limits on the Fourier analysis integral \(-\frac{1}{2}T_0 \) to \( \frac{1}{2}T_0 \), because this gives the simplest form for the answer. If you write down this expression as a mathematical formula, you can do the rest of the lab without the symbolic toolbox. Or, if you don’t have the symbolic toolbox at your disposal, then you will have to derive this mathematical formula by hand—just break the cosine into two complex exponentials and grind out the integrals.

(d) Since \( x(t) \) is periodic, it can be represented approximately by a truncated Fourier series of the form (3) where the Fourier coefficients are given by (2) and the fundamental frequency is \( \omega_0 = 2\pi / T_0 \).

Use your formula for the Fourier coefficients \( a_k \) for \( k = 0, \pm 1, \pm 2, \ldots, \pm N \) to synthesize an approximation to \( x(t) \); use \( N = 10 \) and call the resulting signal \( x_{10}(t) \). This can be done in two different ways: (1) evaluate the numeric values of the Fourier coefficients and use your function \( \text{syn\_fourier( )} \) from an earlier lab, or (2) use the function \( \text{fouriersynth( )} \) if you have the symbolic toolbox.

---

\(^1\) Alternate approach: We can “take the Fourier transform” of the differential equation; the derivative term becomes a multiplication by \( j\omega \) in the frequency domain. and \( H(j\omega) \) is found by dividing \( Y(j\omega) / X(j\omega) \).

\(^2\) You cannot use \( \text{freqz( )} \) because this is not a digital filter.
Make a plot over the range $0 \leq t \leq 3T_0$ that compares the exact $x(t)$ to the approximate one, $x_{10}(t)$. At which times is $x_{10}(t)$ most different from the exact $x(t)$?

5.2 Find the Output Signal

The Fourier coefficients of the output signal are $b_k = a_k H(jk\omega_0)$ because the theory of the frequency response tells us how to determine the exact output of the lowpass filter by tracking each sinusoidal component through the filter: Using our $2N+1$ term approximation for the input, the approximate output is

$$y_N(t) = \sum_{k=0}^{N} b_k e^{jk\omega_0 t} = \sum_{k=0}^{N} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

where the $a_k$ are the Fourier coefficients of $x(t)$.

(a) **Frequency Domain:** Make a three-panel plot showing the spectrum of $x_N(t)$ in the top for $N = 3$; the magnitude of $H(j\omega)$ in the middle (use a frequency range that lines up with the top plot); and the spectrum of $y_N(t)$ in the bottom plot (for $N = 3$ also). To do this, you must have specific values for $R$ and $C$ to determine $\alpha$ and $\beta$ in the formula for $H(j\omega)$, so let $R = 33,000$ ohms and $C = 5 \times 10^{-6}$ farads in this part and the next.

(b) **Time Domain:** Next, you should make a plot of the output signal in the time domain for $N = 10$, i.e., plot $y_{10}(t)$ versus $t$ over the range $0 \leq t \leq 11T_0$.

If you don’t have the symbolic toolbox, then you must evaluate the $b_k$ Fourier coefficients numerically and use `syn_fourier` to create $y_N(t)$. In this approach, the Fourier coefficients $a_k$ must be evaluated numerically, and the frequency response $H(j\omega)$ must be evaluated at the appropriate frequencies. If $\mathbf{H}$ is a vector of frequency response values and $\mathbf{akvals}$ is a vector of Fourier coefficients for the input, then the product would be the Fourier coefficients of the output.

If you have the symbolic toolbox, then to evaluate the output signal in (6), you need to create a symbolic expression for the frequency response $H(j\omega)$ which can be written in terms of symbolic variables for $R$ and $C$. Then substitute into this symbolic expression for $H(j\omega)$ an evaluation at the correct frequencies to create another symbolic function that is a function of $k$. And, finally, multiply by the symbolic expression for the input Fourier coefficients $a_k$. Then you will have a symbolic expression for $b_k$, and you can use `fouriersynth()` to synthesize the output $y_{10}(t)$.

In either case, plot $y_{10}(t)$ versus $t$ over the range $0 \leq t \leq 11T_0$.

5.3 Design the Power Supply in the Frequency Domain

The power supply circuit could be solved in the time-domain to get the output waveform. If you had the signal $y(t)$, then you could measure the DC component of the output voltage, and you would also notice that the output signal contains an oscillating component called the ripple. The objective of the design is to control the size of the ripple by choosing $R$ and $C$ carefully.

(a) Refer to your Fourier Series formula above, and determine the DC value of the input $x(t)$. It is recommended that you mark the DC value of $x(t)$ on the plot(s) of $x(t)$ made previously.

(b) Use your knowledge of the input Fourier series and $H(j\omega)$ to write a mathematical formula for the DC component of the output. Hint: Use the frequency response $H(j\omega)$ to find the DC gain of the first-order RC filter in terms of $R$ and $C$. Also, when you finally get the correct answer for the output DC term, it should not depend on $C$. 

The ripple in the steady-state output is due to all the non-DC terms in the Fourier Series, but it is mostly due to the terms for $k = \pm 1$.

$$y(t) = b_0 + \sum_{k=1}^{\infty} \left( b_k e^{jk\omega_0 t} + b_{-k} e^{-jk\omega_0 t} \right)$$

Therefore, the output can be well approximated by considering the signal $y_1(t)$ that contains only one sinusoidal term plus DC. You might try to derive the mathematical formula for $y_1(t)$ in (6), but you only need the mathematical formula (in terms of $R$ and $C$) for the magnitude of the first sinusoidal term in $y_1(t)$. In the process of doing this derivation you will have to determine the period of the ripple, so give the value of the period in secs.

Now we can complete a general design of the power supply for any specification on the output voltage and ripple. Suppose that we require that the DC component of the output be $V_{\text{out}} = 3.3$ volts with a ripple of $\pm V_r = \pm 0.15$ volts, i.e., peak-to-peak ripple of 0.3 volts. This is the same specifications as in the last question of Lab 14a(Section 4.3(c)), so you already know the answer (albeit approximately). The disadvantage of Lab 14ais that the answer was obtained by “trial and error.”

Now we will solve two non-linear equations in two unknowns to get formulas for $R$ and $C$ so that the DC component of the output is exactly $V_{\text{out}}$ volts, and the oscillating component of $y_1(t)$ satisfies the ripple specification, $\pm V_r$. Use your result from part (c) to set up the two equations: one for DC and the other for $k = 1$. Show that you can solve the DC equation for $R$ in terms of the desired $V_{\text{out}}$. Then plug that result into the second equation and rearrange to get a formula for $C$ in terms of the ripple $\pm V_r$, as well as $R$ and $V_{\text{out}}$. This second equation involves the magnitude of a complex quantity, $H(j\omega)$, so the algebra can be messy.

Verify your formulas in part (c) by calculating the exact values of $R$ and $C$ needed to get $V_{\text{out}} = 3.3$ volts with a ripple of $\pm V_r = \pm 0.15$ volts to see that you get nearly the same answer that you obtained in Lab 14a; the output voltage should ripple between 3.15 and 3.45 volts. Make a plot of the output signal to confirm that your design is correct.
Lab 15
INSTRUCTOR VERIFICATION SHEET
For each verification, be prepared to explain your answer and respond to other related questions that the lab TA’s or professors might ask. Turn this page in at the end of your lab period.

Name: ________________________________  Date of Lab: _________

Part 3.4 Illustrate Fourier synthesis with $a_k$ following a “sinc” formula. Make some plots with the MATLAB function fouriersynth().

Verified:____________________  Date/Time:__________

Part 3.5 Determine the Fourier analysis integral for a half-wave rectified sine. Write out the mathematical formula below. Also give the numerical value of the period. Have your TA check the formula.

Verified:____________________  Date/Time:__________